math up for hacking sdrs

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the plan

hello world

- doing my master in communications engineering in Germany
- hanging around here, FPGAs, uCs, GNU Radio

my life was a lie

let's go back to highschool maths ...

$$
\begin{aligned}
0 &= x^2 + x + 1 \\
x_{1,2} &= -\frac{1}{2} \pm \frac{\sqrt{1^2 - 4}}{2} \\
&= -\frac{1}{2} \pm \frac{\sqrt{-3}}{2}
\end{aligned} \tag{1}
$$

there are *no solutions* to this equation then, right?

my life was a lie (II)

- what if we had something (let's call it j) that

$$\begin{aligned} j^2 &= -1 \\ j &= \sqrt{-1} \end{aligned} \tag{2}$$

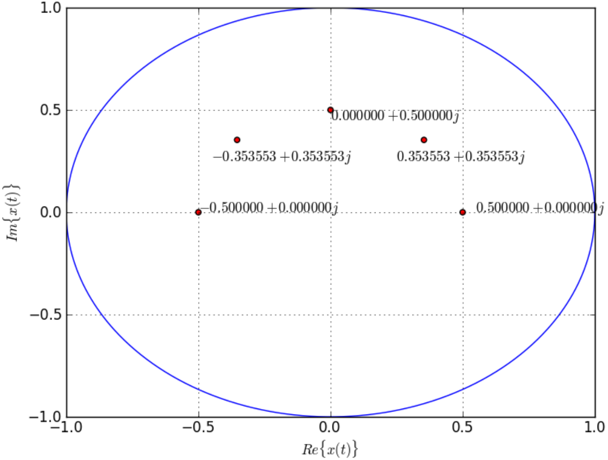
- then we could just write

$$\begin{aligned} x_{1,2} &= -\frac{1}{2} \pm \frac{\sqrt{-1}\sqrt{3}}{2} \\ &= -\frac{1}{2} \pm j\frac{\sqrt{3}}{2} \end{aligned} \tag{3}$$

my life was a lie (III)

- so now our solutions can be written as a sum $x_{1,2} = a \pm jb$.
- we call a the *real* part $\mathfrak{Re}$, and b the *imaginary* part $\mathfrak{Im}$ of $x_{1,2}$.
- let's identify this with a two dimensional vector $\vec{x} = \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \mathfrak{Re}(x) \\ \mathfrak{Im}(x) \end{pmatrix}$

example plot for complex numbers



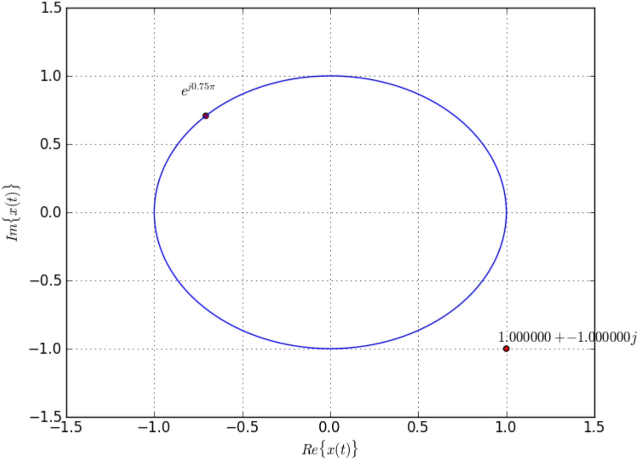
stuff to know about complex numbers

- $x = a + jb = \Re\mathfrak{e}(x) + j\Im\mathfrak{m}(x) = |x|e^{j\phi} = |x|[cos(\phi) + jsin(\phi)]$ (euler's identity)
- $x^* = a - jb = |x|e^{-j\phi}$ (complex conjugate)
- $|x|^2 = xx^* = \Re\mathfrak{e}^2(x) + \Im\mathfrak{m}^2(x) \rightarrow |x| = \sqrt{\Re\mathfrak{e}^2(x) + \Im\mathfrak{m}^2(x)}$
- $j = e^{j\frac{\pi}{2}} = e^{j90^\circ} \rightarrow -j = (j)^* = e^{-j\frac{\pi}{2}} = e^{-j90^\circ}$
- $\phi = tan^{-1}(\frac{\Im\mathfrak{m}(x)}{\Re\mathfrak{e}(x)})$
- in general we prefer radians over degrees, i.e. $\phi_{radian} = \frac{\pi}{180}\phi_{degree}$

some small examples

- what is the real part of $x = 1 - j$?
- what is the imaginary part of $x = 1 - j$?
- what is the absolute value of $x = 1 - j$?
- $\phi(x)$?
- 45° in radians?
- $\frac{6}{8}\pi$ in degrees?

example plot for complex numbers



complex numbers

continuous & discrete vs analog & digital

- analog (continuous in time and value)$\rightarrow x(t) \in \mathbb{R}, \mathbb{C}, t \in \mathbb{R}$
- digital
  - discrete time + float $\rightarrow x(k) \in \mathbb{R}, \mathbb{C}, k \in \mathbb{Z}$
  - discrete time + fixed point $\rightarrow x(k) \in \approx \mathbb{R}, \mathbb{C}, k \in \mathbb{Z}$
  - discrete time + int $\rightarrow x(k) \in \mathbb{Z}, k \in \mathbb{Z}$
  - ...

properties that come to mind

- random, e.g. brownian noise
- deterministic
- periodic $\rightarrow x(t) = x(t + kT_0) \forall k \in \mathbb{Z}$
- real / complex

let's be a bit more systematic

- imagine your signal $x(t)$ as a painting, drawn with a fixed set of watercolors
- kind of coordinates in colors $(0.5red, 1.3blue \dots)$.
- how much of each color has been used to paint this picture?
- step back a bit, abstract, each color is a frequency

fourier transform

$$X(\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt \tag{4}$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(\omega) e^{j\omega t} dt \tag{5}$$

- we call $X(\omega)$ the *spectrum* of x(t)
- fourier transform tells us "how much" of frequency ω is in our signal

example: single tone

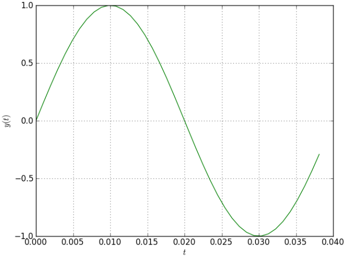


Figure: single tone signal

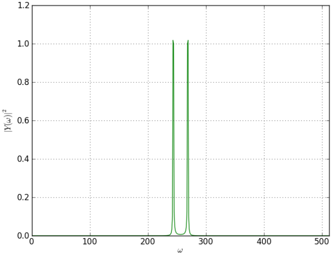


Figure: spectrum $|X(\omega)|^2$

example: dual tone

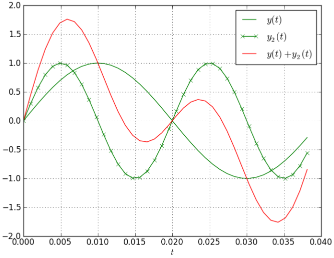


Figure: single tone signal

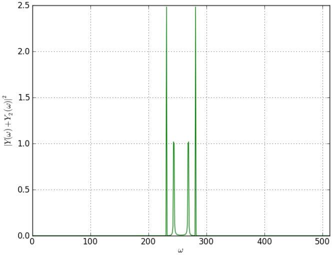


Figure: spectrum $|Y(\omega) + Y_2(\omega)|^2$

properties of the fourier transform

- *linear* (see dual tone example),
  $x_{sum}(t) = \alpha_1 x_1(t) + \alpha_2 x_2(t) \circ\!\!-\!\!\bullet X(\omega)_{sum} = \alpha_1 X_1(\omega) + \alpha_2 X_2(\omega)$
- *translation* $\rightarrow$ phase shift,
  $x(t - t_o) \circ\!\!-\!\!\bullet X(\omega) e^{-j\omega t_o}$
- *scaling*
  $x(at) \circ\!\!-\!\!\bullet \frac{1}{|a|} X(\frac{\omega}{a})$
- *convolution* (explanation later)
  $(f * g)(t) \circ\!\!-\!\!\bullet F(\omega) G(\omega)$
- *uncertainty principle* (same as in qm), i.e. cannot be localized well in time *and* frequency

- goal here to give intuition, if you're interested in the details, check out literature
- we don't calculate this stuff by hand, we use tables of correspondences and the properties from the slide before, is listed e.g. on wikipedia
- interesting correspondences include
  - $1 \circ\!\!-\!\!\bullet 2\pi\delta(x)$
  - $cos(\omega_0 t) \circ\!\!-\!\!\bullet \pi(\delta(\omega - \omega_0) + \delta(\omega + \omega_0))$
  - $e^{j\omega_0 t} \circ\!\!-\!\!\bullet 2\pi\delta(\omega - \omega_0)$
  - $rect(\omega_0 t) = 1 \text{ for } t \in [-\frac{1}{2\omega_0}, \frac{1}{2\omega_0}] \circ\!\!-\!\!\bullet \frac{1}{|\omega_0|} \frac{sin(\frac{\omega}{2\pi\omega_0})}{\frac{\omega}{2\pi\omega_0}}$
- imagine $\delta(x)$ as 1 at x = 0 and 0 everywhere else

example: rectangular pulse

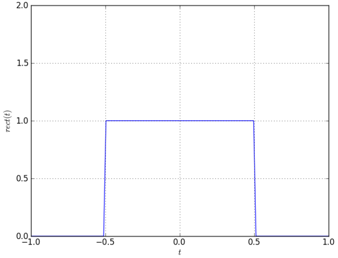


Figure: rectangular pulse, $\omega_0 = 1$

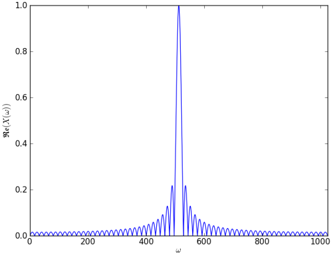


Figure: $\mathfrak{Re}(X(\omega))$

example: rectangular pulse

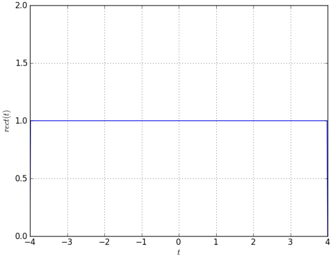


Figure: rectangular pulse, $\omega_0 \approx \frac{1}{8}$

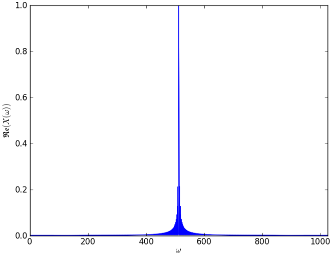
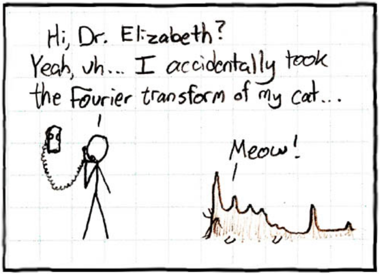


Figure: $\mathfrak{Re}(X(\omega))$

some further remarks



- narrow in {frequency,time} $\rightarrow$ wide in {time,frequency}
- real signals are *always symmetric* w.r.t $\omega = 0$
- use tables!
- I cheated a bit before by using the FFT to calculate the spectra ...

basic idea about fourier transform

system theory

- allows us to classify systems that do something to signals
- here we'll cover so called LTI systems
- again on hand waiving level to gain *intuition*

LTI systems

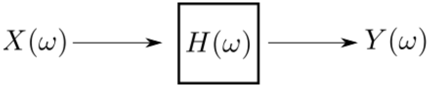
- Linear Time Invariant
- *linear* we already saw that before, remember?
  $S\{\alpha_1 x_1(t) + \alpha_2 x_2(t)\} = \alpha_1 S\{x_1(t)\} + \alpha_2 S\{x_2(t)\}$
- *time invariant* $\rightarrow$ system always behaves the same, i.e. for the same input, we always get the same output.
- why would we limit ourselves to this case?
  - acronyms sound fancy / smart
  - have useful properties
  - huge class of real systems are like this
  - if something is non linear, we can linearize it most of the time

input output relation in an LTI system

- the output is the convolution integral
- $y(t) = \int_{-\infty}^{\infty} x(t)h(t-\tau)d\tau = (x(t) * h(t))(t)$
- wtf is $h(t)$?
- we call $h(t)$ the *impulse response* of a system, i.e. what happens if we plug $\delta(t)$ into the system.
- we could measure h(t)
- obviously convolution integrals are nasty

input output relation in an LTI system (II)

- remember the properties of the fourier transform?
- $(x(t) * h(t))(t) \circ\!\!-\!\!\!-\!\!\bullet X(\omega)H(\omega)$
- $y(t) = (x(t) * h(t))(t) \circ\!\!-\!\!\!-\!\!\bullet Y(\omega) = X(\omega)H(\omega)$
- any idea how to get h(t) now?



input output relation in an LTI system (III)

- system is *completely* described by *either* one of $h(t) \circ\!\!-\!\!\bullet H(\omega)$
- we call $H(\omega)$ a system's *frequency response*
- describes how a system responds to an input signal of frequency ω.
- sine / cosine are *eigenfunctions* of the system

composition of linear systems

- system is *completely* described by *either* one of $h(t) \circ\!\!-\!\!\bullet H(\omega)$
- order of linear operations can be exchanged
- composition can be easily done in fourier domain
- sine / cosine are *eigenfunctions* of the system

$$x(t) \longrightarrow \boxed{S_1\{\circ\}} \longrightarrow \boxed{S_2\{\circ\}} \longrightarrow \boxed{S_3\{\circ\}} \longrightarrow y(t)$$

$$X(\omega) \longrightarrow \boxed{H_1(\omega)} \longrightarrow \boxed{H_2(\omega)} \longrightarrow \boxed{H_3(\omega)} \longrightarrow Y(\omega)$$

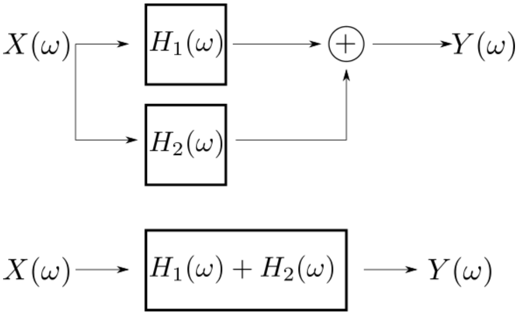
composition of linear systems (II)

$$X(\omega) \longrightarrow \boxed{H_1(\omega)} \longrightarrow \boxed{H_2(\omega)} \longrightarrow \boxed{H_3(\omega)} \longrightarrow Y(\omega)$$

$$X(\omega) \longrightarrow \boxed{H_1(\omega)\, H_2(\omega)\, H_3(\omega)} \longrightarrow Y(\omega)$$

- again note that multiplication is commutative
- so is the application of linear systems

composition of linear systems (III)



- both of the systems are identical
- again note that addition is also commutative
- so is the application of linear systems

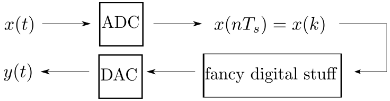
basic idea of LTI systems and composition

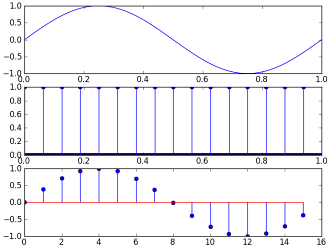
why?

- analog is old ;-)
- quality
- performance
- price

a basic dsp system

- input signal is $x(t)$ which is continouus
- sampling it with a *period* T_s gives us the sequence $x(nT_s) = x(k)$.
- usually we do some digital stuff then
- normally the output will be analog again in some form





how does this translate to math

- input signal is $x(t)$ and continuous
- we want $x(nT)$, so how about $x(nT) = \sum_n x(t)\delta(t - nT)$

$$x(nT_s) \circ\!\!-\!\!\bullet \frac{1}{T} X(\frac{\omega}{T_s})$$

$$x(t + nT_s) \circ\!\!-\!\!\bullet \frac{1}{T} X(\frac{\omega}{T_s}) e^{j\omega \frac{t}{T}}$$

$$\sum_{n=-\infty}^{\infty} x(t + nT_s) \circ\!\!-\!\!\bullet \frac{1}{T_s} \sum_{\omega=-\infty}^{\infty} X(\frac{\omega}{T_s}) e^{j\omega \frac{t}{T_s}} \tag{6}$$

$$\ldots$$

$$\sum_{n=-\infty}^{\infty} x(nT_s)\delta(t - nT_s) \circ\!\!-\!\!\bullet \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X(\omega + \frac{k}{T_s}) \tag{7}$$

second glance at the obtained spectrum

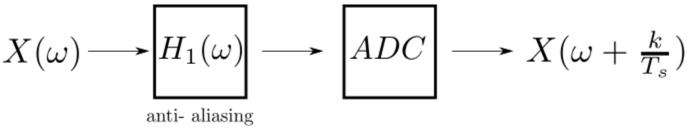
- phew ... quite some formulae there ...
- let's have another look at the last one

$$\sum_{n=-\infty}^{\infty} x(nT_s)\delta(t - nT_s) \circ\!\!-\!\!\bullet \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X(\omega + \frac{k}{T_s}) \tag{8}$$

- spectrum is periodic with $\frac{1}{T_s} = f_s$
- $\rightarrow$ to avoid overlap (aliasing) $|\omega_{max}| \leq 2\pi \frac{f_s}{2}$, i.e. $X(\omega) = 0$ for $|\omega| \notin [0, \frac{f_s}{2}]$

this is important

- we learned we have to sample with $f_s \geq 2f_{max}$ or $|\omega_{max}| \leq 2\pi\frac{f_s}{2}$
- this holds at *all* times if our signal is *bandlimited*
- under the given conditions the spectral content in $[-\frac{f_s}{2}, \frac{f_s}{2}]$ can be used to completely reconstruct x(t)
- there are some exceptions (I'll not discuss however)



sampling theorem

why another transform?

- we just learned how to sample a signal, so now we're digital
- fourier transform and all the system stuff we saw before is analog, though
- using the analog stuff for discrete systems becomes tedious
- specialized transform to handle discrete cases would be nice

the z-transform

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} \tag{9}$$

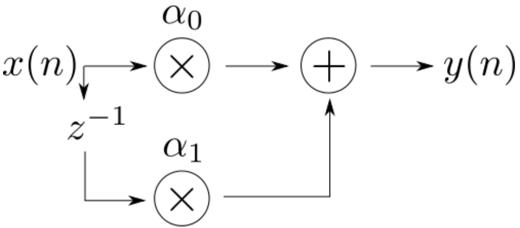
- the inverse is kinda mathy so let's keep this for later
- let's talk a bit about properties
- convergence, i.e. when is $X(z)$ less than ∞?
- linear (we saw that already)

the z-transform (II)

- time lag $x(n-n_0) \circ\!\!-\!\!\bullet X(z)z^{-n_0}$
- convolution $y(n) = \sum_{m=-\infty}^{\infty} x(m)g(n-m) \circ\!\!-\!\!\bullet Y(z) = X(z)G(z)$

the z-transform (III)

- let's throw what we know at an example ;-)



- upper branch: $x(n)\alpha_0 \circ\!\!-\!\!\bullet \alpha_0 X(z)$
- lower branch: $x(n+1)\alpha_1 \circ\!\!-\!\!\bullet \alpha_1 z^{-1} X(z)$
- linear: $y(n) = \alpha_0 x(n) + \alpha_1 x(n+1) \circ\!\!-\!\!\bullet Y(z) = (\alpha_0 + \alpha_1 z^{-1})X(z)$

the z-transform (IV)

- let's have another look at the last part
- $y(n) = \alpha_0 x(n) + \alpha_1 x(n+1) \circ\!\!-\!\!\bullet\, Y(z) = (\alpha_0 + \alpha_1 z^{-1}) X(z)$
- we call $G(z) = \frac{Y(z)}{X(z)}$ the system's *transfer function* in this case $G(z) = (\alpha_0 + \alpha_1 z^{-1})$
- remark: the structure we saw corresponds to a two coefficient digital filter

normalized frequencies

- we saw before that discrete signals are periodic with $\frac{1}{T_s}$
- so we can just look at one period (the first one) from $[\frac{-f_s}{2}, \frac{f_s}{2}]$
- we can identify this with $[-\pi, \pi]$ by just letting $\frac{f_s}{2} = \pi$
- nomenclature not clear in literature usually either Ω or ω for normalized frequencies

the discrete fourier transform

- it might be interesting to see a discrete system's behaviour in frequency
- $X(\Omega) = \sum_{k=0}^{N-1} x(n) e^{-j\Omega n}$
- $x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\Omega) e^{j\Omega n} d\Omega$
- closer look reveals if we let $z = e^{j\Omega}$ we again got our z-transform
- neat: take the transfer function of a system $G(z)$ let $z = e^{j\Omega}$ we got our (discrete) frequency response G($e^{j\Omega}$)

the discrete fourier transform (DFT)

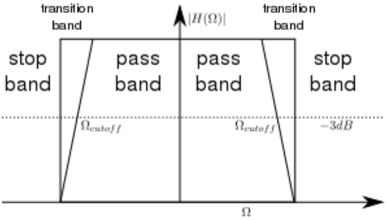
$$X(\Omega) = \sum_{k=0}^{N-1} x(n) e^{-j\Omega n} \tag{10}$$

$$x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\Omega) e^{j\Omega n} d\Omega \tag{11}$$

- gives us the spectrum at N points $\Omega_k = k\frac{2\pi}{N}$ (imagine we divide $[-\pi, \pi]$ into N-1 intervals)
- in these points same as the sampled spectrum obtained by a continuous fourier transform
- assumes implicitly that our signal is periodic
- behaves rather similar to a fourier transform *but* some nasty catches related to periodicity

digital filters

- a system that has (mostly in frequency) a certain behaviour we designed
- one example could be the anti-aliasing lowpass we saw with sampling
  $$H_1(\Omega) = \begin{cases} 1 \text{ if} |\Omega| \leq \pi \\ 0 \text{ otherwise} \end{cases}$$
- on the right kind of prototype lowpass, any idea what might be the problem?



digital filters IIR vs. FIR

- FIR = finite impulse response
- no feedback, always stable
- can be designed to have linear phase
- order = #coefficients -1
- we focus on these ones

- IIR = infinite impulse response
- feedback, may or may not be stable
- cannot have linear phase
- if linear phase is not a requirement, more bang for buck

DIY FIR filters using the windowing method

- technique called window method
- idea: specify ideal frequency response, and calculate it's iverse $\rightarrow$ straightforward
  1. pick your ideal frequency response $H_{ideal}(\Omega)$
  2. pick a filter order (how many coefficients?)
  3. if order is odd we have to cope for the delay to make filter causal $H_{ideal,causal}(\Omega) = H_{ideal}(\Omega)e^{-j\frac{N-1}{2}\Omega}$
  4. compute $h_{ideal,causal}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_{ideal,causal}(\Omega)e^{j\Omega n}d\Omega$
  5. pick window function, calculate $w(n)$ e.g. $w_{hamming}(n) = 0.54 - 0.46cos(\frac{2\pi n}{N-1})$

windowing method example

- say we want to design a lowpass with $\omega_{cutoff} = \frac{\pi}{4}$ and order 4

$$
\begin{aligned}
h_{ideal,causal}(n) &= \frac{1}{2\pi}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} e^{-j\frac{N-1}{2}\Omega} e^{j\Omega n} d\Omega \\
&= [\frac{1}{2\pi(n-\frac{N-1}{2})j} e^{j\Omega(n-\frac{N-1}{2})}]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \\
&= \frac{sin(\frac{\pi}{4}(n-\frac{N-1}{2}))}{\pi(n-\frac{N-1}{2})} \\
&= \frac{sin(\frac{\pi}{4}(n-\frac{4}{2}))}{\pi(n-\frac{4}{2})} \qquad (12) \\
w(n) &= 0.54 - 0.46cos(\frac{2\pi n}{4}) \qquad (13)
\end{aligned}
$$

numpy and gnuradio

- numpy's fft goes from $[0, 2\pi]$ not from $[-\pi, \pi]$ so use `numpy.fft.fftshift()`
- matplotlib + pylab is quite nice
- `ipython -pylab` or `bpython` make quite nice IDEs
- when working with real world stuff avoid designing filters by hand ;-)